Yiddish of Day

gebentsh zol zayn
der shtern fun velkhn
es gist der shvits

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געבענטש זאָל זײַן
דער שטערן פֿון װעלכן
עס גיסט דער שװיץ

may the forehead from
which the sweat falls off
be blessed

Complexification

<u>Idea</u>: $\mathbb{R}$ = ☹

$\mathbb{C}$ = ☺

$\rightsquigarrow$ turn $\underline{\mathbb{R}}$-vs into $\underline{\mathbb{C}}$-vs

<u>Recall</u>: $\mathbb{C}$ is a $\underline{\mathbb{R}}$-vector space (of dim $\underline{2}$)

Def: Let V be $\mathbb{R}$-vs. Then the complexification of V denoted $V_{\mathbb{C}}$ is $\mathbb{C} \otimes V$

Prop: V a $\mathbb{R}$-vs with basis $B = (v_1, \dots, v_n)$. Then

1) $V_{\mathbb{C}}$ is a $\mathbb{C}$-vs (with scalar mult $\alpha \cdot (z \otimes v) = \alpha z \otimes v$)

2) A $\mathbb{C}$ basis of V is $B_{\mathbb{C}} = (1 \otimes v_1, 1 \otimes v_2, \dots, 1 \otimes v_n)$

(ie $\dim_{\mathbb{R}} V = \dim_{\mathbb{C}} V_{\mathbb{C}}$)

Pf) For (1) use that $- \otimes -$ is bilinear (over $\mathbb{R}$) and that all the field axioms over $\mathbb{C}$ can give us the axioms needed.

(2) Take $\sum \left(c_i(z_i \otimes v_i')\right)$ with $c_i, z_i \in \mathbb{C}$ and $v_i' \in V$

Then $\sum c_i(z_i \otimes v_i') = \sum c_i z_i \otimes v_i'$

$= \sum c_i z_i (1 \otimes v_i')$

Now write $v_i' = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$

Plug in to get that the original tensor is of form

$$\sum d_i (1 \otimes v_i)$$

So $(1 \otimes v_i)$ spans.

To prove linear independence, Define the map

$$\mathbb{C}^n \longrightarrow V_{\mathbb{C}}$$

$$e_1 \longrightarrow 1\otimes v_1$$

$$e_2 \longrightarrow 1\otimes v_2$$

$$\vdots$$

$$e_n \longrightarrow 1\otimes v_n$$

This is injective because $1\otimes v_i \neq 1\otimes v_j$ for $i \neq j$.

(exc.) By the HW therefore this set

$(1\otimes v_1, \dots, 1\otimes v_n)$ is LI ex) i) $V = \mathbb{R}^n \rightsquigarrow V_{\mathbb{C}} = \mathbb{C} \otimes \mathbb{R}^n \xrightarrow{\sim} \underline{\mathbb{C}^n}$

$(\lambda \otimes \vec{v}) \mapsto \lambda \vec{v}$

$(1 \otimes v_i) \to e_i$

ii) $V = M_{m \times n}(\mathbb{R}) \rightsquigarrow V_{\mathbb{C}} = \mathbb{C} \otimes M_{m \times n}(\mathbb{R}) \xrightarrow{\sim} M_{m \times n}(\mathbb{C})$ $\quad \lambda \times m \to \lambda m$

iii) $V = \mathbb{R}[t]_{\leq n} \rightsquigarrow V_{\mathbb{C}} = \mathbb{C} \otimes \mathbb{R}[t]_{\leq n} \xrightarrow{\sim} \mathbb{C}[t]_{\leq n}$

Warning: If we view $V = \mathbb{C}$ as a $\mathbb{R}$-vs then

$V_{\mathbb{C}} \neq \underline{\mathbb{C}}$

(HW)

The complexification is more than just a $\mathbb{C}$ vs constructed out of an $\mathbb{R}$-vs. It is "the best" (universal) one

Thm: V be an $\mathbb{R}$-vs and W a $\mathbb{C}$-vs.

Suppose $f: V \to W$ is an $\mathbb{R}$-linear map.

Then $\exists!$ $\mathbb{C}$-linear map $\tilde{f}: V_{\mathbb{C}} \to W$

such that

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "="] & W \\ V_{\mathbb{C}} \arrow[ur, "\exists!\,\tilde{f}"'] & \end{tikzcd}$$

$V \to V_{\mathbb{C}}$

$v \mapsto 1 \otimes v$

Now let $\varphi: V \to V'$ be linear map of $\mathbb{R}$-vs.

Then by HW, have linear map

$$\underline{\varphi_{\mathbb{C}}}: \underline{\mathbb{C}} \otimes_{\mathbb{R}} V \longrightarrow \underline{\mathbb{C}} \otimes V'$$

$$\begin{array}{ccc} \underline{\mathbb{C}} \otimes V & & \underline{\mathbb{C}} \otimes V' \\ \| & & \| \\ \underline{V_{\mathbb{C}}} & & \underline{V'_{\mathbb{C}}} \end{array}$$

(Denote $\underline{\varphi_{\mathbb{C}}} := \underline{1 \otimes \varphi}$)

Turns out this is complex-linear too!

Thm: Keeping the notation above, let

$\mathcal{B}_V = (v_1 \dots v_n)$ basis for V, V' $\longrightarrow$ $\mathcal{B}_{V_{\mathbb{C}}}$ complexified basis

$\mathcal{B}_{V'} = (w_1 \dots w_m)$ $\quad$ $\mathcal{B}_{V'_{\mathbb{C}}}$

Then $[\varphi]_{\mathcal{B}_V}^{\mathcal{B}_{V'}} = \underline{[\varphi_{\mathbb{C}}]_{\mathcal{B}_{V_{\mathbb{C}}}}^{\mathcal{B}_{V'_{\mathbb{C}}}}}$

Pf) Lets denote $\varphi(v_i) = \sum^m c_j w_j$

Then $\varphi_{\mathbb{C}}(1 \otimes v_i) = 1 \otimes \varphi(v_i)$

$= 1 \otimes \sum c_j w_j$

$$= \sum c_i (1 \otimes w_i)$$

So the columns of the matrix are the same!

[$n=2$ $m=2$

$$g(v_1) = c_{11}w_1 + c_{12}w_2$$

$$g(v_2) = c_{21}w_1 + c_{22}w_2$$

$$\rightsquigarrow g_c(1 \otimes v_1) = 1 \otimes g(v_1) = 1 \otimes (c_{11}w_1 + c_{12}w_2)$$

$$= c_{11}\, 1 \otimes w_1 + c_{12}\, 1 \otimes w_2$$

$$g_c(1 \otimes v_2) = 1 \otimes g(v_2) = 1 \otimes (c_{21}w_1 + c_{22}w_2)$$

$$= c_{21}\, 1 \otimes w_1 + c_{22}\, 1 \otimes w_2$$]

[in words, can think of a real matrix as already "being" a complex matrix]

Cor: $g: V \to V'$ linear of $\mathbb{R}$-vs. Then

i) $(\operatorname{Ker} g)_{\mathbb{C}} = \operatorname{Ker}(g_{\mathbb{C}})$

ii) g injective $\iff$ $g_{\mathbb{C}}$ is injective

iii) $(\operatorname{im} g)_{\mathbb{C}} = \operatorname{im}(g_{\mathbb{C}})$

iv) g surjective $\iff$ $g_{\mathbb{C}}$ is surjective

Conjugation:

Recall the function $\overline{(\ )} : \mathbb{C} \to \mathbb{C}$

$$z \to \bar{z}$$

$\bar{z} = a - ib$

$z = a + ib$

- then note that, for $r \in \mathbb{R}$

$$(\bar{r}) = r$$

On the other hand if $(\bar{z}) = z$

then $z \in \mathbb{R}$

$\rightsquigarrow$ that is we can recover $\mathbb{R}$ as the "fixed points" of $\overline{(-)}$

<u>Def</u>: V be $\mathbb{R}$-vs. Define the map

$$\tau : \underline{V_{\mathbb{C}}} \longrightarrow \underline{V_{\mathbb{C}}}$$

by $\tau(z \otimes v) = \underline{\bar{z} \otimes v}$

We call this map the standard <u>Conjugation</u> on $V_{\mathbb{C}}$

HW) Show the following commutes

$$\begin{tikzcd} \mathbb{C} \otimes \mathbb{R}^n \arrow[r, "\sim"] \arrow[d, "\tau"] & \mathbb{C}^n \arrow[d, "\overline{(\ )}"] \\ \mathbb{C} \otimes \mathbb{R}^n \arrow[r, "\sim"] & \mathbb{C}^n \end{tikzcd}$$

(That is, this "recovers" the standard conjugation on $\mathbb{C}^n$)

Prop: In the set-up above, the fixed points of

$\tau : \underline{V_{\mathbb{C}}} \longrightarrow \underline{V_{\mathbb{C}}}$

is the subspace $\underline{\{ r \otimes v \mid r \in \mathbb{R} \} \simeq V}$

Pf) Suppose $\alpha \otimes v$ is fixed under τ.

Then $\tau(\alpha \otimes v) = \bar{\alpha} \otimes v = \alpha \otimes v$

$= \bar{\alpha}(1 \otimes v) = \alpha(1 \otimes v)$

$\Rightarrow \bar{\alpha} = \alpha$

OTOH if $r \in \mathbb{R}$ then

$$\tau(r \otimes v) = \bar{r} \otimes v = r \otimes v$$

Now define the map

$$\left\{ r \otimes v \;\middle|\; \begin{array}{l} r \in \mathbb{R} \\ v \in V \end{array} \right\} \longrightarrow V$$

$$r \otimes v \longmapsto rv$$